Dual Reciprocity Boundary Element Analysis for the Laminar Forced Heat Convection Problem in Concentric Annulus

Chang-Yong Choi*

(Received May 18, 1998)

This paper presents a study of the dual reciprocity boundary element method (DRBEM) for the laminar heat convection problem in a concentric annulus with constant heat flux boundary condition. DRBEM is one of the most successful technique used to transform the domain integrals arising from the nonhomogeneous term of the Poisson equation into equivalent boundary only integrals. This recently developed and highly efficient numerical method is tested for the solution accuracy of the fluid flow and heat transfer study in a concentric annulus. Since their exact solutions are available, DRBEM solutions are verified with different number of boundary element discretizations and internal points. The results obtained in this study are discussed with the relative error percentage of velocity and temperature solutions, and potential applicability of the method for the more complicated heat convection problems with arbitrary duct geometries.

Key Words: Dual Reciprocity Boundary Element Method, Concentric Annulus, Laminar Heat Convection, Heat Flux Boundary Condition, Numerical Method

Nomenclature

- b : Heat source-like term
- c : Constant
- f : Interpolating function
- G : Coefficient matrix involving q^*
- H : Coefficient matrix involving u^*
- L : Number of internal points
- N : Number of boundary elements
- n : Normal unit vector
- p : Pressure
- Q : Matrix involving q
- q : Normal derivative of u
- $\bar{q}$: Specified q value
- $\breve{q}$: Normal derivative of $\breve{u}$
- r : Radius or radial distance
- T : Temperature
- U : Matrix involving u
- u : General dependent variable
- $\bar{u}$: Specified u value
- $\breve{u}$: Particular solution variable
- w : Axial flow velocity

Greek Letters

- α : Thermal diffusivity or coefficient of interpolating function
- μ : Dynamic viscosity
- Γ : Boundary
- Ω : Domain
- ϕ : Linear interpolating function
- θ : Angle

Subscripts

- i : Source point or inner boundary
- j : Collocation point
- k : Element number
- m : Mean value
- o : Outer boundary
- w : Wall surface

Superscripts

- $*$: Fundamental solution

1. Introduction

Among the various numerical methods, the boundary element method (BEM) becomes one

* Department of Mechanical Engineering Jeonju University, Jeonju 560-759, Korea

of the favorite analysis tool ever since its introduction to the solution of heat transfer problems. Its advantage over the finite difference or the finite element methods comes from the fact that instead of full domain discretization, only the boundary is discretized into elements and internal point position can be freely defined. Therefore the quantity of data necessary to solve the problems can be greatly reduced (Brebbia et al., 1984).

Until recent years the main area of the BEM application has been limited to the conduction heat transfer problems among different heat transfer modes and therefore, with various research efforts, BEM for the solution of heat conduction direct or inverse problem is now well established (Kane, 1994; Choi and Kim, 1996; Choi, 1997). However BEM study for the application of heat convection problems can be considered as insufficient and in still developing stage. Since the convection effects are of considerable importance in many heat transfer phenomena, they need much more research focus. The main difficulties of the BEM application to such problems are due to the facts that the fundamental solutions are only available for the few governing equation types and, except Laplace equation, additional domain discretization is required to account source type domain integral terms (Skerget and Rek, 1995).

The dual reciprocity boundary element method (DRBEM) which was introduced by Nardini and Brebbia (1982) is thus far the most successful technique for dealing with above mentioned lack of fundamental solution types and domain integral problems. Since its introduction DRBEM has been applied in many field of engineering problems (Partridge et al., 1992; Partridge, 1995; Wrobel and DeFigueiredo, 1991). In the DRBEM, available fundamental solution is used for the complete governing equation, and domain integral arising from the heat source-like term is tranferred to the boundary by using the radial basis interpolation functions (Partridge, 1994; Yamada et al., 1994).

This paper presents the application of DRBEM to the Poisson type equations, and fully developed laminar convection heat transfer problems in concentric annulus are illustrated as their applicable examples. The concentric annulus is chosen because of its simplicity and available exact solutions, so that basic nature of the proposed method for the convection problems can be analyzed and revealed in a detailed manner (Kakac et al., 1987; Kays and Crawford, 1993). Therefore present research efforts are confined within basic study aiming at the establishment of DRBEM's applicability for the heat convection analysis to be eventually extended in the future study of various heat transfer system.

In this paper, hydrodynamically and thermally fully developed laminar flow with uniform heat flux through thermal boundary in a concentric annulus is studied by using the DRBEM. To verify the methods on heat convection problems, numerical solutions with different number of boundary element discretizations and internal points are compared with the exact solutions for its convergence and accuracy.

2. Formulation of the Problem

Consider an incompressible Newtonian fluid flow in a concentric annular tube as shown in

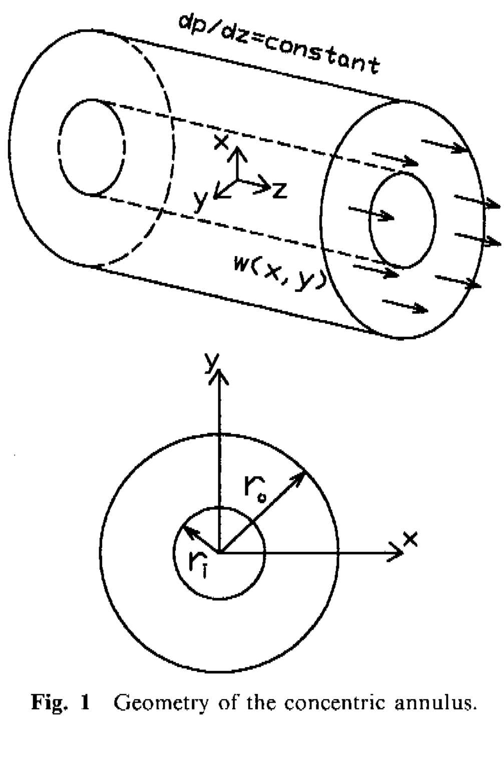

Fig. 1 Geometry of the concentric annulus.

Fig. 1. In the system to be analyzed, z coordinate represents the axial direction and x-y coordinates are attached to the cross-sectional surface. The inner and outer cylinder radii are taken as r_i and r_o. For the fully developed steady laminar flow with constant transport properties and negligible body forces, Navier-Stokes equation becomes simple pressure-driven Poiseuille flow equation. Since the flow is fully developed, axial flow velocity is a function of only x-y coordinates, and axial pressure gradient is constant. In the energy equation, the viscous dissipation and axial heat conduction effects are neglected. Therfore the governing equation can be expressed in the form of a Poisson equation as follows:

Momentum equation:

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dz} \qquad (1)$$

Energy equation:

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{w}{a}\frac{\partial T}{\partial z} \qquad (2)$$

In these equations, w, μ, p, T, and $a(=k/\rho c)$ have their usual meaning of axial flow velocity, coefficient of viscosity, pressure, temperature and thermal diffusivity, respectively. For the thermally fully developed flow with constant heat flux boundary condition, Eq. (2) can be rewritten by using the mixed mean temperature T_m (Kays and Crawford, 1993) as

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{w}{a}\frac{dT_m}{dz} \qquad (3)$$

where $\frac{\partial T}{\partial z} = \frac{dT_m}{dz} = constant$ from the given conditions. The boundary conditions associated with the Eqs. (1) and (3) are

$$w=0 \ at \ r=r_i, \ w=0 \ at \ r=r_o \qquad (4)$$
$$T=T_i \ at \ r=r_i, \ T=T_o \ at \ r=r_o \qquad (5)$$

where subscripts i and o represent for the inner and outer surfaces.

For the solution of temperatures, velocity from Eq. (1) is obtained first and then Eq. (3) can be solved in sequence since the assumption of constant viscosity uncoupled the momentum and energy equations.

3. Dual Reciprocity Boundary Element Equation

For the BEM solution, Eqs. (1) and (3) subject to Eqs. (4) and (5) can be generalized as the following type of Poisson equation (Partridge et al., 1992).

$$\nabla^2 u(x, y) = b(x, y), \qquad (x, y) \in \Omega \quad (6)$$

with the boundary conditions:

$$u(x, y) = \bar{u}, \qquad (x, y) \in \Gamma_1 \quad (7)$$
$$q(x, y) = \frac{\partial u(x, y)}{\partial n} = \bar{q}, \quad (x, y) \in \Gamma_2 \quad (8)$$

and to represent convective heat transfer problems:

$$u(x, y) = w, \quad b(x, y) = \frac{1}{\mu}\frac{dp}{dz} = constant$$
(See Eq. (1))
$$u(x, y) = T, \quad b(x, y) = \frac{w}{a}\frac{dT_m}{dz}$$
(See Eq. (3))

where $\Gamma_1 + \Gamma_2 = \Gamma$ is the total boundary of solution domain Ω, n is the normal to the boundary and $\bar{u}$ and $\bar{q}$ denote the specified values at each boundary.

Applying the usual boundary element technique to Eq. (6), an integral equation can be deduced as follows (Brebbia et al., 1984).

$$c_i u_i + \int_\Gamma u q^* d\Gamma - \int_\Gamma q u^* d\Gamma = \int_\Omega b u^* d\Omega \quad (9)$$

where the constant c_i depends on the geometry at point i as

$$c_i = \begin{cases} 1 & \text{for } (x_i, y_i) \in \Omega \\ \frac{\theta}{2\pi} & \text{for } (x_i, y_i) \in \Gamma \end{cases} \qquad (10)$$

where θ is the angle between the tangent to Γ on either side of point i.

The key method of DRM is to take the domain integral of Eq. (9) to the boundary and remove the needs of complicated domain discretization. To accomplish this idea, the source term $b(x, y)$ is expanded as its values at each node j and a set of interpolating functions f_j are used as (Partridge et al., 1992; Partridge, 1995; Wrobel and DeFigueiredo, 1991)

$$b(x,y)\simeq\sum_{j=1}^{N+L}a_j f_j \tag{11}$$

where the a_j is a set of initially unknown coefficients and $N+L$ is the number of boundary nodes plus internal points. If the function $\hat{u}_j$ can be found such that

$$\nabla^2\hat{u}_j=f_j \tag{12}$$

then the domain integral can be transferred to the boundary.

Substituting Eq. (12) into Eq. (11), and applying integration by parts twice for the domain integral term of Eq. (9) leads to

$$c_i u_i+\int_\Gamma uq^* d\Gamma-\int_\Gamma qu^* d\Gamma=\sum_{j=1}^{N+L}a_j\Big(c_i\hat{u}_{ij}+\int_\Gamma\hat{u}_j q^* d\Gamma-\int_\Gamma\hat{q}_j u^* d\Gamma\Big) \tag{13}$$

For the two-dimensional domain of interest in this study, u^*, q^* and $\hat{u}$, $\hat{q}$ can be derived as (Partridge et al., 1992)

$$u^*=\frac{1}{2\pi}\ln\Big(\frac{1}{r}\Big) \tag{14}$$

$$q^*=\frac{-1}{2\pi r}\nabla r\cdot\vec{n}$$

$$\hat{u}=\frac{r^2}{4}+\frac{r^3}{9} \tag{15}$$

$$\hat{q}=\Big(\frac{r}{2}+\frac{r^2}{3}\Big)\nabla r\cdot\vec{n}$$

where r stands for the distance from a source point i or a DRM collocation point j to a field point (x, y). As for the Eq. (12), a radial basis function $f=1+r$ is chosen as an interpolating function which was shown to be generally sufficient (Partridge, 1994; Partridge, 1995; Yamada et al., 1994).

In the numerical solution of the integral Eq. (13), u, q, $\hat{u}$ and $\hat{q}$ in the integrals are modelled using the linear interpolation functions as

$$\int_{\Gamma_k}uq^* d\Gamma=u_k h_{ik}^1+u_{k+1}h_{ik}^2 \tag{16}$$

$$\int_{\Gamma_k}qu^* d\Gamma=q_k g_{ik}^1+q_{k+1}g_{ik}^2 \tag{17}$$

$$\int_{\Gamma_k}\hat{u}_j q^* d\Gamma=\hat{u}_{kj}h_{ik}^1+\hat{u}_{(k+1)j}h_{ik}^2 \tag{18}$$

$$\int_{\Gamma_k}\hat{q}_j u^* d\Gamma=\hat{q}_{kj}g_{ik}^1+\hat{q}_{(k+1)j}g_{ik}^2 \tag{19}$$

where

$$h_{ik}^1=\int_{\Gamma_k}\phi_1 q^* d\Gamma,\quad h_{ik}^2=\int_{\Gamma_k}\phi_2 q^* d\Gamma \tag{20}$$

$$g_{ik}^1=\int_{\Gamma_k}\phi_1 u^* d\Gamma,\quad g_{ik}^2=\int_{\Gamma_k}\phi_2 u^* d\Gamma \tag{21}$$

Here the first subscript of Eq. (20) and (21) refers to the specific position of the point where the flow velocity or temperature is evaluated; the second subscript refers to the boundary element over which the contour integral is carried out. The superscript 1 and 2 designate the linear interpolation function ϕ_1 and ϕ_2 respectively, with which the u^* and q^* functions are weighted in the integrals in Eq. (16) through (19).

For the boundary $\Gamma=\Gamma_1\cup\Gamma_2$ discretized into N elements, integral terms in Eq. (13) can be rewritten as

$$\int_\Gamma uq^* d\Gamma=\sum_{k=1}^{N}\int_{\Gamma_k}uq^* d\Gamma=\sum_{k=1}^{N}[h_{i(k-1)}^2+h_{ik}^1]u_k$$
$$=\sum_{k=1}^{N}H_{ik}u_k\quad or=\sum_{j=1}^{Nn}H_{ik}\hat{u}_{kj}\ \text{for}\ \hat{u}_j \tag{22}$$

$$\int_\Gamma qu^* d\Gamma=\sum_{k=1}^{N}\int_{\Gamma_k}qu^* d\Gamma=\sum_{k=1}^{N}[g_{i(k-1)}^2+g_{ik}^1]q_k$$
$$=\sum_{k=1}^{N}G_{ik}q_k\quad or=\sum_{j=1}^{Nn}G_{ik}\hat{q}_{kj}\ \text{for}\ \hat{q}_j \tag{23}$$

where $h_{i0}^2=h_{iN}^2$ and $g_{i0}^2=g_{iN}^2$. Introducing Eq. (22) and (23) into Eq. (13) and manipulating results yields a dual reciprocity boundary element equation as

$$c_i u_i+\sum_{k=1}^{N}H_{ik}u_k-\sum_{k=1}^{N}G_{ik}q_k=\sum_{j=1}^{N+L}a_j\Big(c_i\hat{u}_{ij}\sum_{k=1}^{N}H_{ik}\hat{u}_{kj}-\sum_{k=1}^{N}G_{ik}\hat{q}_{kj}\Big) \tag{24}$$

4. Numerical Solution

For the computer implementation of numerical solution, Eq. (24) can now be written in a matrix form as

$$HU-GQ=(H\hat{U}-G\hat{Q})\,a \tag{25}$$

where H and G are matrices of their elements being H_{ik} and G_{ik}, with c_i being incoroperated into the principal diagonal element, respectively. U, Q and their terms with hat of Eq. (25) correspond to vectors of u_k, q_k and matrices with jth column vectors of hat u_{kj}, q_{kj}. It is noted that vector a of unknown coefficients j can be evaluated from Eq. (11) with chosen interpolating function f_j and the function $b(x, y)$ of governing equation. Therefore introducing the boundary conditions into the nodes of u_k and q_k vectors and rearranging by taking known quantities to the right hand side and unknowns to the left hand side leads to a set of simultaneous linear equations of the form

$$Ax = B \tag{26}$$

Using the DRBEM matrix equation, the numerical solution of laminar convection heat transfer problem in a concentric annulus can be readily obtained as x being the flow velocity w for momentum equation and also temperature T for energy equation or their normal derivatives, respectively.

Consider the geometry illustrated in Fig. 2. For the sake of simplification, the surface temperatures of two cylinders are assumed to be equal. Thus, the solution satisfies the following boundary conditions

$$w(x, y)|_{r=r_i} = w(x, y)|_{r=r_o} = 0,$$
$$T^*(x, y)|_{r=r_i} = T^*(x, y)|_{r=r_o} = 0 \tag{27}$$

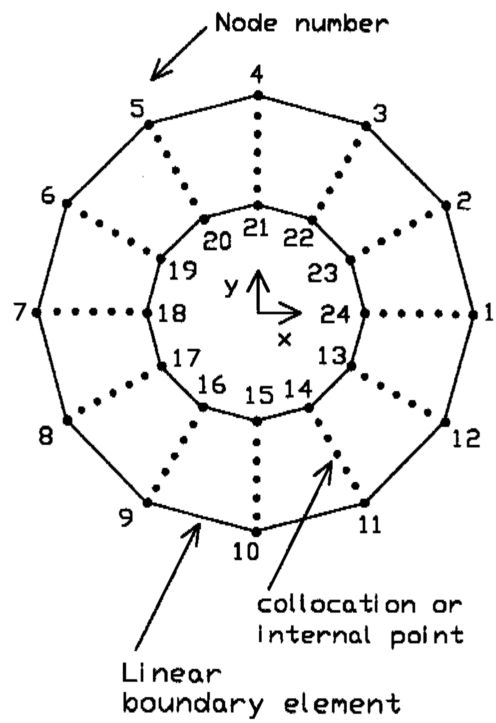

Fig. 2 Boundary element nodes and internal points for the system to be analyzed.

and $T^* = T - T_w$, $T_w = T_i = T_o$

As a note, no slip conditions are applied for the velocity boundary condition.

For the numerical test case, following numerical values in Eq. (1) and (3) are taken from the paper (Sim and Kim, 1996) where the spectral collocation method is used for the eccentric annuli heat convection anlysis.

$$r_i = 0.05\ m, \quad r_o = 0{,}025\ m$$
$$\frac{1}{\mu}\frac{dp}{dz} = -837.99\ (m \cdot \text{sec})^{-1}$$
$$\alpha = 1.342 \times 10^{-9}\ m^2/\text{sec}$$
$$\frac{dT_m}{dz} = 0.5\ ^\circ\text{C}/m$$

5. Results and Discussion

In order to confirm the accuracy of the dual reciprocity boundary element method for the present heat convection problem, each boundary of outer and inner surface is equally discretized as 12, 18, 24, 36 and 48 elements respectively. As for the number of internal or DRM collocation nodes, 5 nodes across the pair of inner and outer boundary element nodal points are located as shown in Fig. 2. Therefore total number of internal points used in the analysis are 60, 90, 120, 180 and 240 for each 12, 18, 24, 36 and 48 boundary element cases respectively.

To obtain the axial flow velocity distribution $w(x, y)$, Eq. (1) is solved first. Their results for the boundary and internal nodes are shown in

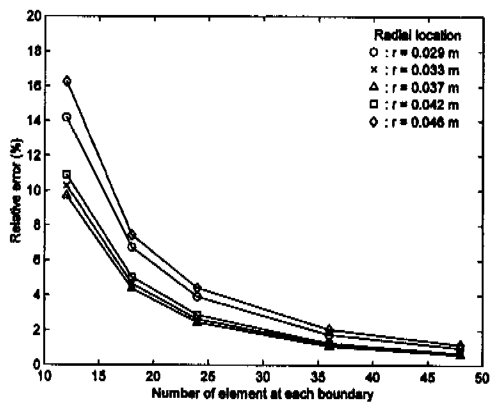

Fig. 3 Accuracy test for the velocity solution at the selected internal points.

Table 1 DRBEM results with exact solutions for the boundary and internal locations in flow velocity analysis.

Solution Variable	Radial Location (r)	DRBEM Solution (number of boundary elements case)					Exact Solution
		12	18	24	36	48	
$\partial w/\partial n$	0.050	-8.7914	-9.2518	-9.4117	-9.5253	-9.5648	-9.61571
$\partial w/\partial n$	0.025	-12.8556	-12.4808	-12.3530	-12.2634	-12.2324	-12.19320
w	0.029	0.045760	0.042787	0.041643	0.040784	0.040476	0.040077
w	0.033	0.067479	0.064041	0.062800	0.061907	0.061594	0.061190
w	0.037	0.072513	0.068996	0.067733	0.066829	0.066513	0.066108
w	0.042	0.062889	0.059576	0.058339	0.057435	0.057118	0.056710
w	0.046	0.039932	0.036911	0.035864	0.035054	0.034749	0.034346

Table 2 DRBEM results with exact solutions for the boundary and internal locations in temperature analysis ($T^*=T_w-T$).

Solution Variable	Radial Location (r)	DRBEM Solution (number of boundary elements case)					Exact Solution
		12	18	24	36	48	
$\partial T^*/\partial n$	0.050	182861.2	181225.9	180421.1	179821.5	179669.9	181363.4
$\partial T^*/\partial n$	0.025	255844.9	251946.1	250847.8	250063.7	249795.8	251872.1
T^*	0.029	980.08	943.45	932.53	925.09	922.61	923.73
T^*	0.033	1558.05	1512.78	1500.47	1491.94	1488.83	1488.79
T^*	0.037	1699.77	1647.50	1633.81	1624.86	1621.71	1621.76
T^*	0.042	1417.04	1368.80	1353.28	1344.21	1341.61	1342.60
T^*	0.046	1261.09	780.38	753.08	748.90	746.48	748.05

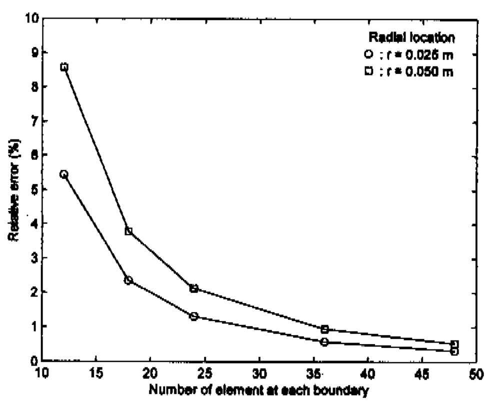

Fig. 4 Accuracy test for the normal derivative of velocity solution at the inner and outer boundaries.

Table 1. Here the normal derivative of velocity w at the boundary is listed as well, and all the numerical solutions are compared with the exact solutions (Kays and Crawford, 1993) for their accuracy. Figures. 3 and 4 show the convergence plot of DRBEM velocity and its normal derivative solutions as the number of boundary elements and internal points increase. DRBEM solutions are in close agreement with the exact solutions and relative errors are within 5% for the above 24 element cases. As noted in Fig. 3, velocity solutions at location of $r=0.046$ and $r=0.029$ are less accurate than the others and, in between, $r=0.046$ point gives more inaccurate solution than $r=0.029$. And for the normal derivatives of velocity at

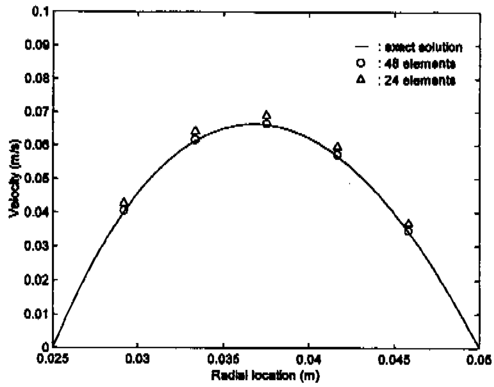

Fig. 5 Velocity profile of exact solution compared with DRBEM results.

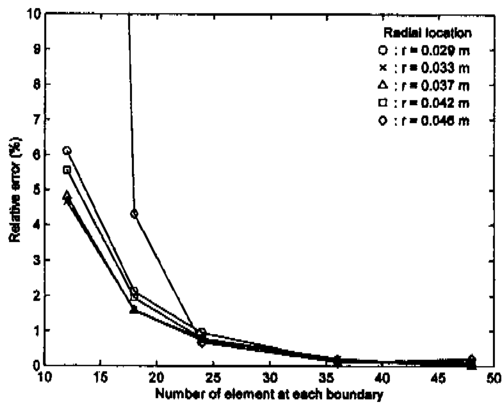

Fig. 6 Accuracy test for the temperature solution at the selected internal points.

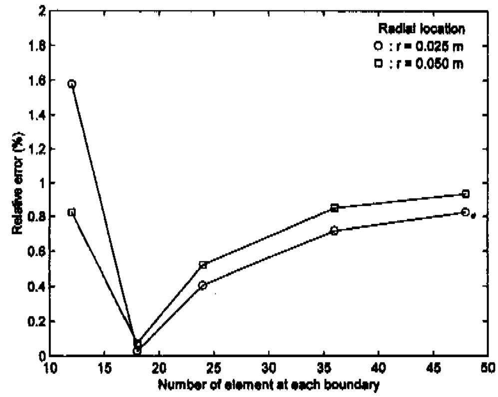

Fig. 7 Accuracy test for the normal derivative of temperature solution at the inner and outer boundaries.

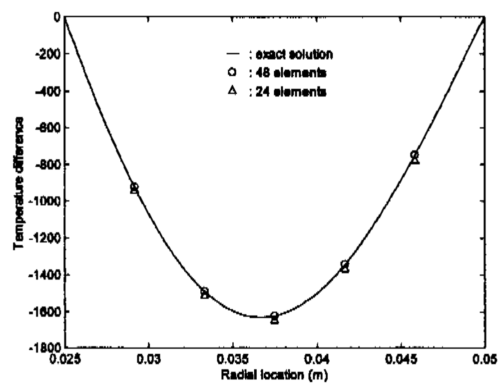

Fig. 8 Temperature profile ($T^*=T-T_w$) of exact solution compared with DRBEM results.

boundary $r=0.05$ is less accurate than $r=0.025$ as shown in Fig. 4. These results are due to the facts that the outer boundary element size is larger than the inner boundary element size and distribution of internal points is getting sparse to the outward direction, whereas rapid change of velocity occurs at inner and outer boundary sides as illustrated in Fig. 2 and 5. Therefore solution's error magnitude regarding to the radial location is closely related to both the physical and the mathematical aspects and nevertheless overall solution accuracy is shown to be fairly acceptable. However 12 element solution case shows maximum 16.3% error at radial position $r=0.046$ and later results in very inaccurate temperature solution.

Then these DRM velocity solutions are, in turn, used in the energy equation (Eq. (3)) to solve for the temperature distribution. Table 2 shows the results, and it is found that DRM solutions are in excellent agreement with exact solutions and relative errors are within 1% for the above 24 element cases (see Fig. 6 and 7). Although the converging trend in Fig. 7 is not monotonic and radial location effect about error magnitude is not exactly following the previously discussed velocity solution case, solution trends can be considered as indistinguishable within 1% relative error. These test results validate the power of dual reciprocity boundary element method and its solution accuracy, since the numerically solved velocity was used as an input in Eq. (3) and source-like function $b(x, y)$ of Eq. (11) in Eq. (3) is approximated with interpolating function and nodal values of internal points. As a final note, the 12 element case turns out to be inadequate for the solution of this probelm. Error of the velocity solution is amplified, and unacceptable temperature solution error of 68.5% is observed at radial location r=0.046.

6. Conclusions

A dual reciprocity boundary element method has been presented for the solution of laminar heat convection problem in a concentric annulus imposed with constant heat flux. DRBEM matrix is formulated to perform the numerical implementation, and five cases of boundary element discretization are tested with the corresponding number of internal points. Five radial locations are selected to obtain the velocity and temperature solutions. Test results are shown to be in excellent agreement with exact solutions for the above 24 element case. However 12 element case results in inaccurate solution and errors are shown to be amplified in solving the energy equation. As a final remark, recently developed dual reciprocity boundary element method is successfully applied to solving the laminar heat convection problem in a concentric annulus, and also current study shows its broad potentiality for further applications.

References

Brebbia, C. A., Telles, J. C. F. and Wrobel, L. C., 1984, *Boundary Element Techniques*, Springer-Verlag, New York.

Choi, C. Y. and Kim, H. G., 1996, "Boundary Element Solutions of the Geometrical Inverse Heat Conduction Problems Using Regularization Procedure," *Proceedings of the 2nd European Thermal-Sciences and 14th UIT National Heat Transfer Conferences*, Vol. 2, pp. 1089~1095.

Choi, C. Y., 1997, "Detection of Cavities by Inverse Heat Conduction Boundary Element Method Using Minimal Energy Technique," *Journal of the Korean Society for Nondestructive Testing*, Vol. 17, No. 4, pp. 237~247.

Kakac, S., Shah, R. K. and Aung, W., 1987, *Handbook of Single-Phase Convective Heat Transfer*, John Wiley, New York.

Kane, J. H., 1994, *Boundary Element Analysis in Engineering Continuum Mechanics*, Prentice-Hall, New Jersey.

Kays, W. M. and Crawford, M. E., 1993, *Convective Heat and Mass Transfer*, McGraw-Hill, Singapore.

Nardini, D. and Brebbia, C. A., 1982, "A New Approach for Free Vibration Analysis Using Boundary Elements," *Boundary Element Methods in Engineering*, Compuataional Mechanics Publications, Southamption, pp. 312~326.

Partridge, P. W., Brebbia, C. A. and Wrobel, L. C., 1992, *The Dual Reciprocity Boundary Element Method*, Computational Mechanics Publications, Southamption.

Partridge, P. W., 1994, "Dual Reciprocity BEM: Local versus Global Approximation Functions for Diffusion, Convection and Other Problems," *Engineering Analysis with Boundary Elements*, Vol. 14, pp. 349~356.

Partridge, P. W., 1995, "New Developments in the Dual Reciprocity Method," *Boundary Elements XVII*, Computational Mechanics Publications, Southamption, pp. 11~18.

Sim, W. G. and Kim, J. M., 1996, "Application of Spectral Collocation Method to Conduction and Laminar Forced Heat Convection in Eccectric Annuli," *KSME Journal*, Vol. 10, No. 1, pp. 94~104.

Skerget, L. and Rek, Z., 1995, "Boundary-Domain Integral Method Using a Velocity-Vorticity Formulation," *Engineering Analysis with Boundary Elements*, Vol. 15, pp. 359~370.

Wrobel, L. C. and DeFigueiredo, D. B., 1991, "A Dual Reciprocity Boundary Element Formulation for Convection-Diffusion Problems with Variable Velocity Fields," *Engineering Analysis with Boundary Elements*, Vol. 8, pp. 312~319.

Yamada, T., Wrobel, L. C. and Power, H., 1994, "On the Convergence of the Dual Reciprocity Boundary Element Method," *Engineering Analysis with Boundary Elements*, Vol. 13, pp. 291~298.